1st Dec

Calculus and Analytical Geometry

3rd Dec

Complex Numbers.

A number of the form $x+iy$ where $x, y \in R$. where x is real part and y is imaginary part.

eq. $2+3i \Rightarrow (2, 3)$

MCQs

⇒ C.N does not hold order properties.

⇒ Every real number is a complex number with 0 as its imaginary part.

$15 + 0i$

Properties of complex Numbers.

Addition. ⇒ $(3,5) + (4,6)$

Multiplication ⇒ $(3,-1)\cdot(5,2)$

$(3-i)(5+2i) \Rightarrow 15+6i-5i-2i^2$

Division. $(3,2) \div (1,2)$.

$= 17+i \Rightarrow (17,1)$

$= \frac{3+2i}{1+2i} \times \frac{1-2i}{1-2i}$ .....

Conjugate Complex Numbers.

If we have $x+iy$ then conjugate will be $x-iy$

⇒ Every Real Number is self conjugate.

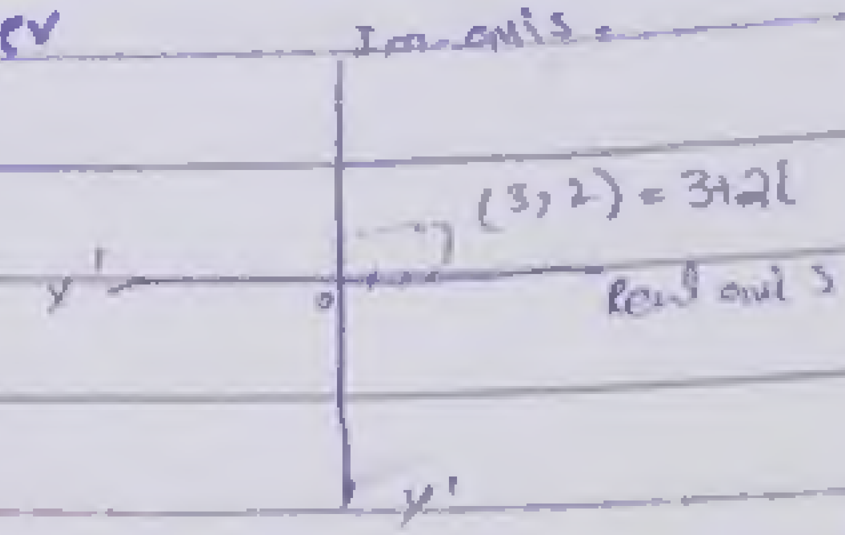
eg= $z = 5$

$\bar{z} = 5$

2nd lec

Calculas and analytical Geometry

Geometrical Interpretation of complex number



Modulus of complex number.

complex numbers a origion se distance modulus of complex number.

$z = x + iy$

$|z| = \sqrt{x^2+y^2}$

eg $z = 5+6i \Rightarrow |z| \sqrt{(5)^2+(6)^2}$

$= \sqrt{61}$

Polar form of complex number

$x+yi = r\cos\theta + ir\sin\theta$

$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1} y/x$

eg $z = 1+i\sqrt{3}$

$x = 1, \quad y = \sqrt{3}$

$1+i\sqrt{3} = r\cos\theta + ir\sin\theta \rightarrow ①$

$r = \sqrt{(1)^2+(\sqrt{3})^2}, \quad \theta = \tan^{-1}\frac{\sqrt{3}}{1}$

$\sqrt{4}, \quad \theta = 60°$

$r = 2 \quad , \quad \theta = 60^\circ$

$1 + i\sqrt{3} = 2\cos 60^\circ + i\, 2\sin 60^\circ$

De Moivers Theorem (Imp)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \forall n \in \mathbb{Z}$

Application Simplify $(\sqrt{3} + i)^3$

firstly in polar form.

$\sqrt{3} + i = r\cos\theta + ir\sin\theta$

$x = \sqrt{3} \; , \; y = 1.$

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

$\theta = \tan^{-1} y/x \Rightarrow \tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow 30^\circ$

$\sqrt{3} + i = 2\cos 30^\circ + 2i\sin 30^\circ$

Now applying De-Moivers theorm.

$= (2\cos 30^\circ + 2i\sin 30^\circ)^3$

$= (2(\cos 30^\circ + i\sin 30^\circ))^3$

$= 8(\cos 30^\circ + i\sin 30^\circ)^3.$

$= 8(\cos 90^\circ + i\sin 90^\circ)$

$= 8(0 + i(1))$

$= 8i$

$= 0 + 8i$

$= (0, 8)$

$(1-\sqrt{3}i)^5$

$1-\sqrt{3}i = r\cos\theta + ri\sin\theta$

$x = 1 \quad , \quad y = -\sqrt{3}$

$r = \sqrt{(1)^2+(-\sqrt{3})^2} \quad , \quad \theta = \tan^{-1}\frac{-\sqrt{3}}{1}$

$r = \sqrt{1+3}$

$r = \sqrt{4}$

$r = 2 \quad , \quad r = -60^\circ$

$1-\sqrt{3}i = 2\cos(-60^\circ) + 2i\sin(-60^\circ)$

Now apply De-moivers law.

$(1-\sqrt{3}i)^5 = (2\cos(-60^\circ) + i2\sin(-60^\circ))^5$

$= (2(\cos(-60^\circ) + i\sin(-60^\circ)))^5$

$= 32(\cos(-60^\circ) + i\sin(-60^\circ))^5$

$= 32(\cos(-300^\circ) + i\sin(-300^\circ))$

$= 32\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$(1+\sqrt{3}i)^5$

$1+\sqrt{3}i = r\cos\theta + ir\sin\theta$

$x = 1 \quad , \quad y = \sqrt{3}$

$r = \sqrt{(1)^2+(\sqrt{3})^2} \quad , \quad \theta = \tan^{-1}\frac{\sqrt{3}}{1}$

$r = \sqrt{4} \quad , \quad \theta = \tan^{-1}\frac{1}{\sqrt{3}}$

$r = 2 \quad , \quad \theta = 60^\circ$

$1+\sqrt{3}i = 2\cos 60^\circ + i\,2\sin 60^\circ$

Now apply De-moivers theorm

$= (2\cos 60^\circ + i2\sin 60^\circ)^5$

$= (2(\cos 60^\circ + i\sin 60^\circ))^5$

$= 32(\cos 300^\circ + i\sin 300^\circ)$

$= 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

Proof of De-moivers theoram.

Mathemadical Induction

(i) Proof the given statement for $n=1$.

(ii) Assume it is true for $n=k$

(iii) Proof that it is true for $n=k+1$.

$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$.

$\cos(A-B) =$

$\sin(A-B) =$

Statement: if $n \in Z$ then.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

proof for $n=1$.

$L.H.S = (\cos\theta + i\sin\theta)^1 \Rightarrow \cos\theta + i\sin\theta$.

$R.H.S = \cos 1\theta + i\sin 1\theta \Rightarrow \cos\theta + i\sin\theta$.

Hence it is true for $n=1$.

Sir Waqar

Computer

Assume that it is true for $n=k$

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \longrightarrow (1)$$

Now for $n=k+1$.

$$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$$

L.H.S $= (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)^1$

By equation (1)

$= (\cos k\theta + i\sin k\theta)\cdot(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) - \sin k\theta\sin\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k\theta+\theta) + i(\sin(k\theta+\theta))$

$= \cos(k+1)\theta + i(k+1)\theta$

$=$ R.H.S

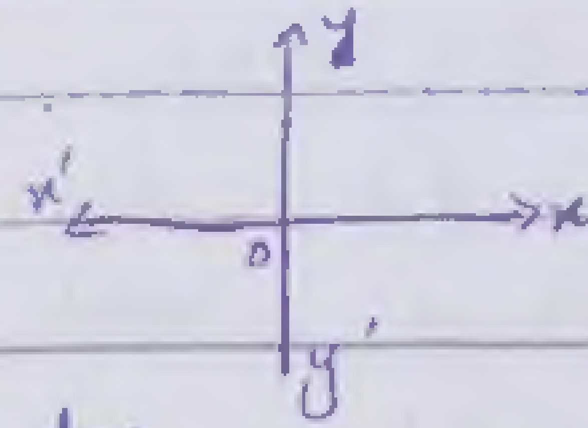
Assignment

simplify $\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}\right)^3$

By using De-Moverics theorem.

Simple Cartesian Curves

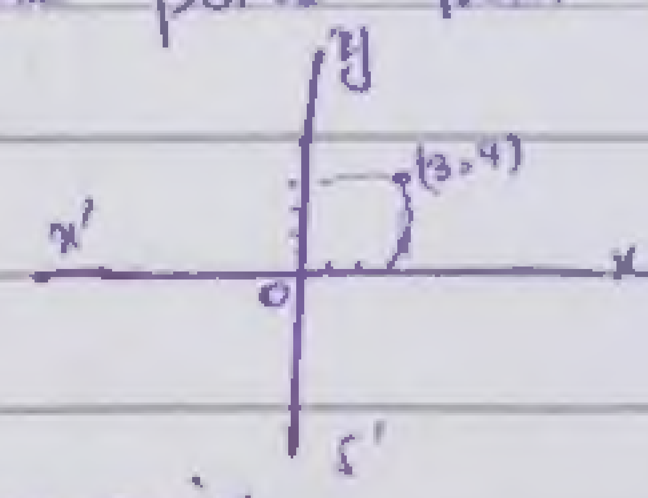
⇒ Cartesian plane



⇒ Cartesian Coordinates

Cartesian coordinates actually describe the distance of the point from origion ~~distance~~

(3, 4)



- always draw in order pair.

⇒ Cartesian Product

$A = \{1, 2, 3\}$

$B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c),$
$(3, a), (3, b), (3, c)\}$

$B \times A = \ldots\ldots ?$

if only one statement is given to find cartesian product such as

$A = \{a, b, c\}$

then

$A \times A = \ldots\ldots$

3rd Dec

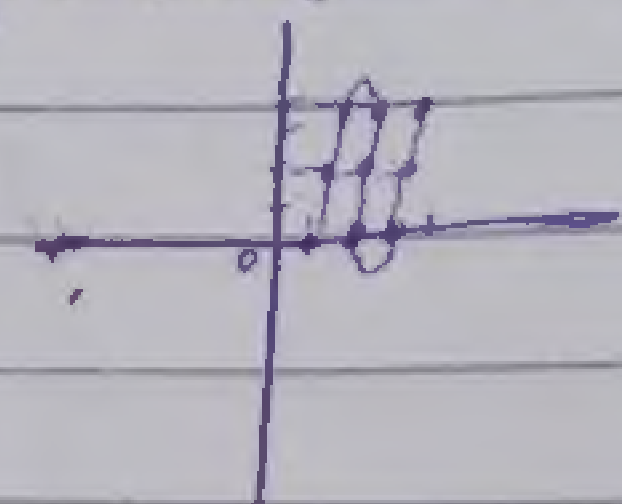
Graph of Cartesian Product

$A = \{1, 2, 3\}$

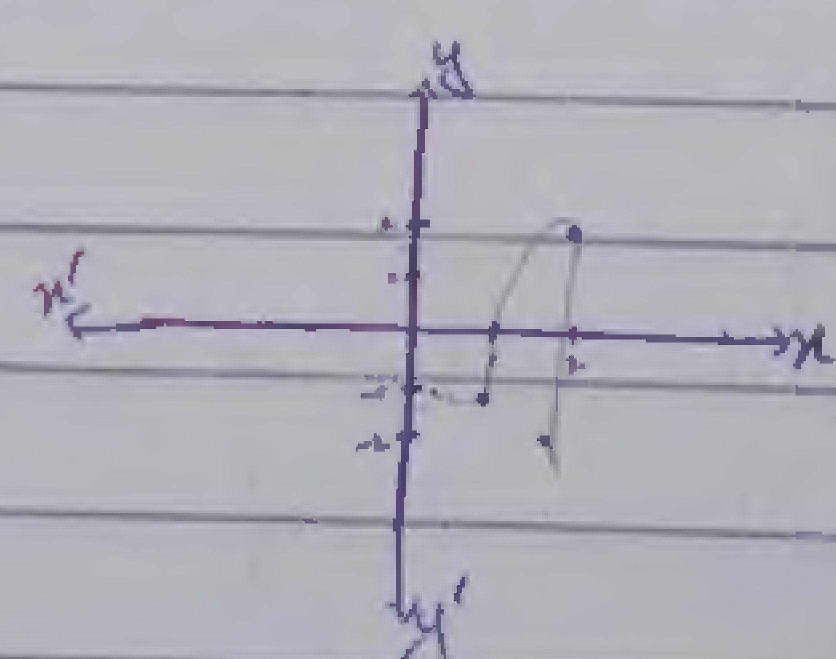
$B = \{0, 2, 4\}$

Graph of cartesian product A and B

$A \times B = \{(1,0), (1,2), (1,4), (2,0), (2,2),$
$(2,4), (3,0), (3,2), (3,4)\}$.



Show that Graph of cartesien plane $\{(+2, 2), (1,-1), (+2,-2)\}$ is a curve or straight line.



cartesian product of any sets can be represented by cartesian diagram we can plot the order pairs in by taking the first element along x-axis and second along y-axis in plane.
Each order pair is marked by point

Calculus and Analytical Geometry

Types of Simple Cartesian Curves.

1. Straight line

An equation of first degree in x and y is an equation of the form.

$Ax+By+C=0.$

where A, B and C are constants.

eg $3x+2y+5=0$

Some special cases of straight line

(1) The slope intercept form.

$$y=mx+c$$

m is slop and c is y-Interсept.

$$m=\tan\alpha$$

$$m=\frac{y_2-y_1}{x_2-x_1}, \quad m=-\frac{a}{b}$$

(2) Two intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$

(3) Normal form

$$x\cos\alpha+y\sin\alpha=p$$

(4) Point slope form.

$$y-y_1=m(x-x_1)$$

(5) Two point form

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

(6) Parametric form $\dfrac{x-a}{\cos\alpha}=\dfrac{y-b}{\sin\alpha}$

(7) Eq of straight line passes through point of intersection of two lines

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$

- Horizontal line

  parallel to x-axis

  (m) slop = 0

- Vertical line.

  Parallel to y-axis

  $m = \infty$

- if two line are parallel $\parallel$ (slope equal)

  $m_2 = m_1$

- if two lines are perpendicular

  $m_1 m_2 = -1.$

Examples

l_1 $2x + y - 4 = 0$

l_2 $x - 5y - 1 = 0$

l_3 $6x + 8y - 3 = 0$

l_4 $4x - 3y - 5 = 0$

(i) write down an equation of straight line parallel to l_1 and passing throog point (2,1)

slope of $l_1 = -\frac{2}{1} = -2$

Slope of req line $= -2$ $\because$ lines are parallel

Equation of required [illegible]

[illegible] and [illegible]

Using point slope form

$y - y_1 = m(x - x_1)$

$y - 1 = -2(x - 2)$

$y - 1 = -2x + 4$

$2x + y - 5 = 0$

(ii) Write [illegible] straight line which

is [illegible] and [illegible] through (1,2)

[illegible]

[illegible] $= -1$ [illegible]

[illegible] $= -1$

[illegible] $= -1$

[illegible]

[illegible] line through (1,2)

[illegible] slope -5

$y - y_1 = m(x - x_1)$

$y - 2 = -5(x - 1)$

y [illegible] $= -5x + 5$

[illegible] $= 0$

Write [illegible] straight [illegible]

intersection [illegible]

[illegible]

[illegible] $) + k(a_2x_1 + b_2y_1 + c_2) =$ [illegible]

[illegible]

at $(2, 3)$

[illegible] $) + k($ [illegible] $5(3) - 1) = 0$

[illegible]

$3 - 14k = 0$

$14k = -3$

circle topic

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

I we

I do

Do I

Ayesh

Ayesha

Does

You

Yes

Calculus and Analytical Geometry

Function $y = f(x)$

Area of square $= l \times w$

$A = x \times x \Rightarrow x^2$

[illegible]

$F(t) = 2(t-1)+3$

Find f at $0, 2, x+2$

$f(0) = 2(0-1)+3$

$= -2+3 \Rightarrow 1$

$F(2) = 2(2-1)+3$

$= 5$

$f(x+2) = 2(x+2-1)+3$

$= 2x+2+3 \Rightarrow 2x+5$

Domain

Interval: closed $[6, 10] = 6, 7, 8, 9, 10$

open $(6, 10) = 7, 8, 9$

$(6, 10] = 7, 8, 9, 10$

[illegible]

Example

* [illegible]

omain. $\mathbb{R}$ [illegible]

Range [illegible]

* [illegible]

[illegible] $\mathbb{R}$

Range [illegible]

* $y = \sqrt{x}$

Domain = [illegible]

Range = [illegible]

* $y = \sqrt{1 - x^2}$

Domain = [illegible]

Range = [illegible]

[illegible]

[illegible]

$\Rightarrow f(t) = \frac{1}{\sqrt{t}}$

Domain: $[1, \infty)$
Range = $[1, \infty)$

$\Rightarrow f(x) = \sqrt{4-(x)^2}$

Domain: $[-2, 2]$
Range = $[0, 2]$

Programming Fundamentals

Calculus and Analytical Geometry

Graph of a function.

Graph of a function is actually a graph of equation $y = f(x)$. and it consist of the points (x, y)

=> Not every curve represent a function

=> **Vertical line test**

To analyse a curve we draw a vertical line if it intersects more then one point then it is not a graph of function.

Procedure

Step 1 Make table of xy pair that specifies the function

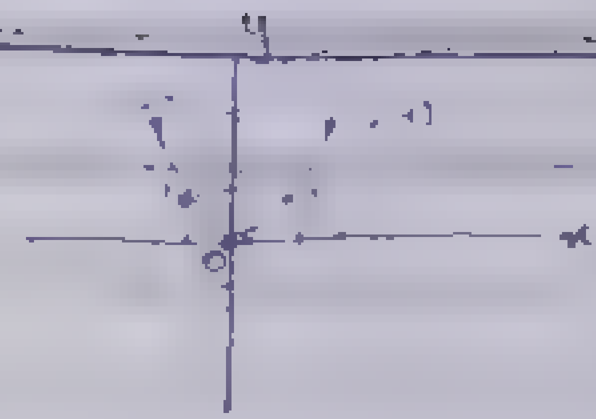
Step 2 plot that pairs in cartesian plane.

Step 3 Join these points to draw the graph

Examples:-

Graph the function $y = x^2$ over the interval $[2, 2]$.

x	x^2
2	4
1	1
0	0
-2	4



<u>Assignment</u> $y = \sqrt{x}$ on $[-4, 4]$

$y = e^x$ $[-2, 2]$

Even and odd functions

Even: if $f(-x) = f(x)$.

Odd: if $f(-x) = -f(x)$

Example:-

$f(x) = 3$

$f(-x) = 3$

neither Even nor odd

• $y = \frac{1}{x^2}$

$f(x) = \frac{1}{x^2}$

$f(-x) = \frac{1}{(-x)^2}$

$= \frac{1}{x^2}$ Even

• $f(x) = \frac{1}{|x|}$

Replace $x = -x$.

$f(-x) = \frac{1}{|-x|}$

$f(x) = \frac{1}{|x|}$

Even

• $f(x) = \frac{x}{x^2-1}$

$\frac{-x}{(-x)^2-1} = \frac{-x}{x^2-1}$

odd

made negative to positive

★ Piecewise defined function.

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|2| = 2$

$|-3| = -(-3) = 3$

$f(x) = \begin{cases} 3x+5 & \text{if } [illegible] \\ x^2-1 & \text{if } 6 < x \le [illegible] \end{cases}$

find value of $f(x)$ at $x = 3$

$f(3) = 3(3)+5 = 14$

at $x = 8$

$(8)^2 - 1 = 63$

Limits of functions

$f(x) = \frac{1}{x-1}$

at $x = 1$

- Rules for finding the limits

⇒ Properties of limits

1) Sum Rule

$\lim_{x \to c} [f(x) + g(x)] = L+M$

eg $\lim_{x \to 3} (x^2+5x+3)$

$= (3)^2 + 5(3) + 4) = 27$

2) Difference Rule

3) Product Rule

4) Constant Multiple Rule

$\lim_{x \to 2} kf(x) = kL$

$= \lim_{x \to 2} 2(x^2-5)$

$= 2((2)^2-5) = -2$

5) Quotient Rule

6) Power Rule $\lim_{x \to 5} (x-2)^5 \Rightarrow (5-2)^5 \Rightarrow 3^5$

Example

[illegible]

$= \lim_{x \to -2} \sqrt{4x^2 - 3}$

[illegible]

$= \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$

Exp $\frac{x^2 + x - 2}{x^2 - 1}$

$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1}$

$\frac{(1)^2 + 1 - 2}{1^2 - 1} = \frac{0}{0}$ (Indeterminate) (meaningless)

By eleminating its denominator algebrically

$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

$\lim_{x \to 1} \frac{x^2 + 2x - x - 2}{x^2 - x} \Rightarrow \lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$

~~Apply limit~~

$= \frac{1+2}{1} = 3$

Exp $\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$

$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \times \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}}$

$\lim_{h \to 0} \frac{(\sqrt{2+h})^2 - (\sqrt{2})^2}{h(\sqrt{2+h} + \sqrt{2})} \Rightarrow \lim_{h \to 0} \frac{\cancel{2} + \cancel{h} - \cancel{2}}{\cancel{h}(\sqrt{2+h} + \sqrt{2})}$

$= \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{\sqrt{2+0} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$

Exp $\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16} \Rightarrow \lim_{v \to 2} \frac{v^3 - (2)^3}{(v^2)^2 - (4)^2}$

$\lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v^2-4)(v^2+4)}$

$\lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)}$

Apply limit

$\frac{4+2(2)+4}{(2+2)(4+4)} = \frac{12}{32} = \frac{3}{8}$

Exp

$\lim_{x \to 0} f(x) = 1$

$\lim_{x \to 0} g(x) = -5$

Evaluate $\lim_{x \to 0} \frac{2f(x) - g(x)}{(f(x)+7)^{2/3}}$

$= \frac{\lim_{x \to 0} (2f(x) - g(x))}{\lim_{x \to 0} (f(x)+7)^{2/3}}$

$=$ Apply limit

$\rightarrow \frac{2(1)-(-5)}{(1+7)^{2/3}} \Rightarrow \frac{2+5}{(8)^{2/3}} \Rightarrow \frac{7}{4}$

Exp $\lim_{h \to 0} \frac{5}{\sqrt{5h+4}+2}$

By Rationalization

$\lim_{x\to 0}$ [illegible]

$\Rightarrow \frac{5x(\sqrt{5x+4}+2)}{5x+4-4}$

[illegible]

Exp $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$\lim_{x\to -1} \frac{(\sqrt{x^2+8})^2-(3)^2}{(x+1)(\sqrt{x^2+8}+3)}$$

$$\lim_{x\to -1} \frac{x^2+8-9}{(x+1)\sqrt{x^2+8}+3} \Rightarrow \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$\lim_{x\to -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} \Rightarrow \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3}$$

Apply limit

$$\frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3} = \frac{2}{6}$$

Exp $\lim \frac{x-1}{\sqrt{x+3}-2}$

Calculus and Analytical Geometry

Limit of a function:-

Exp:- $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$

$=\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{(1-x)(1+x+x^2)}\right)$

$=\lim_{x\to 1}\left(\frac{1+x+x^2-3}{(1-x)(1+x+x^2)}\right)$

$=\lim_{x\to 1}\left(\frac{x^2+x-2}{(1-x)(1+x+x^2)}\right)$

$=\lim_{x\to 1}\left(\frac{x^2+2x-x-2}{(1-x)(1+x+x^2)}\right)$

$=\lim_{x\to 1}\left(\frac{(x-1)(x+2)}{(1-x)(1+x+x^2)}\right)$

$=\lim_{x\to 1}\left(\frac{-(1-x)(x+2)}{(1-x)(1+x+x^2)}\right)$

= Apply limit

$=\frac{(1+2)}{1+1+(1)^2} \Rightarrow -\frac{3}{3} \Rightarrow -1$

$\because a^2-b^2 = (a+b)(a-b)$

Exp:- $\lim_{y\to x}\frac{y^{2/3}-x^{2/3}}{y-x}$

$=\lim_{y\to x}\left(\frac{(y^{1/3})^2-(x^{1/3})^2}{y-x}\right)$

$=\lim_{y\to x}\left(\frac{(y^{1/3}-x^{1/3})(y^{1/3}+x^{1/3})}{y-x}\right)$

x^2+4x-

$x(x+y$

Geometry

$$= \lim_{y \to x} \frac{(y^{1/3} - x^{1/3})(y^{1/3} + x^{1/3})}{(y^{1/3})^3 - (x^{1/3})^3}$$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$= \lim_{y \to x} \frac{\cancel{(y^{1/3} - x^{1/3})}(y^{1/3} + x^{1/3})}{\cancel{(y^{1/3} - x^{1/3})}(y^{2/3} + y^{1/3}x^{1/3} + x^{2/3})}$$

Apply limit

$$= \frac{x^{1/3} + x^{1/3}}{x^{2/3} + x^{1/3} \cdot x^{1/3} + x^{2/3}} \Rightarrow \frac{2x^{1/3}}{2x^{2/3} + x^{2/3}}$$

$x^{1/3} \cdot x^{1/3}$
Base same
Power add
$x^{1/3 + 1/3}$
$= x^{2/3}$

$$= \frac{2x^{1/3}}{3x^{2/3}} \Rightarrow \frac{2}{3} x^{1/3} \cdot x^{-2/3}$$

$$= \frac{2}{3} x^{-1/3} \Rightarrow \frac{2}{3x^{1/3}}$$

Exp: $\lim_{x \to -1} \frac{x^{1/3} + 1}{x + 1}$

$$\lim_{x \to -1} \frac{x^{1/3} + 1}{(x^{1/3})^3 + (1)^3} \Rightarrow \lim_{x \to -1} \frac{\cancel{x^{1/3} + 1}}{\cancel{(x^{1/3} + 1)}(x^{2/3} - x^{1/3} + 1)}$$

$$= \lim_{x \to -1} \frac{1}{x^{2/3} - x^{1/3} + 1}$$

$$= \frac{1}{(-1)^{2/3} - (-1)^{1/3} + 1} \Rightarrow \frac{1}{((-1)^2)^{1/3} - (-1)^{1/3} + 1}$$

$$= \frac{1}{1 - (-1)^{1/3} + 1} = \frac{1}{2 - (-1)^{1/3}}$$

$x^2 + 4x - 2x - 8$
$x(x+4) - 2(x+4)$

Exp: $\lim_{x \to -2} \frac{x^2 + 2x - 8}{x^2 - 4}$

$$\lim_{x \to -2} \frac{x^2 + 4x - 2x - 8}{(x \quad 2)(x - 2)}$$

$$\lim_{x \to -2} \frac{x(x+1) \cdot (x+4)}{(x-2)(x+2)}$$

$$\lim_{x \to -2} \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x+2)}$$

Apply limit

$$\frac{-2+4}{2+2} = \infty$$

Limits of Piecewise functions:-

Exp

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$$

Find limit at $x = 1$

$$L.H.L = \lim_{x \to 1} (x^2)$$

$$= (1)^2 \to 1.$$

$$R.H.L = \lim_{x \to 1} (x^3)$$

$$1 \to 1$$

L.H.S = R.H.S.

$$\lim_{x \to 1} f(x) = f$$

Exp

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ 4x^2 & \text{if } x > -1 \end{cases}$$

Find limit at $x = -1$.

$$L.H.S = \lim_{x \to -1} (x+2)$$

$$-1+2 = 1$$

$$R.H.S = \lim_{x \to -1} (4x^2)$$

$= a(-1)^2 = a$

Exp:- $f(x) = \begin{cases} 2 & \text{if } x \le -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Find limit at $x = -2$ and $x = 2$

at $x = -2$

L.H.L $\lim_{x \to -2^-} (3)$

$= 3$

R.H.L $\lim_{x \to -2^+} (-\frac{1}{2}x^2)$

$= -\frac{1}{2}(-2)^2$

$= -2$

L.H.L $\neq$ R.H.L

So limit doesnot exist at $x=-2$.

At $x = 2$

L.H.L. $\lim_{x \to 2^-} (-\frac{1}{2}x^2)$

$= -\frac{1}{2}(2)^2$

$= -2$

R.H.L. $\lim_{x \to 2^+} (3)$

$= 3$

L.H.L $\neq$ R.H.L

Limit doesnot exist at $x=2$

Absolute value function:-

$\Rightarrow |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

$\Rightarrow |x+3| = \begin{cases} x+3 & \text{if } x+3 > 0 \text{ or } x > -3 \\ -(x+3) & \text{if } x+3 < 0 \text{ or } x < \end{cases}$

Exp:- $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{1}{|x-3|} \right)$

$= \lim_{x \to 3} \left(\frac{1}{x-3} - \frac{1}{(x-3)} \right)$

$= \lim_{x \to 3} \left(\frac{1}{x-3} + \frac{1}{x-3} \right)$

$= \lim_{x \to 3} \left(\frac{1+1}{x-3} \right)$

$= \lim_{x \to 3} \left(\frac{2}{x-3} \right)$

apply limit

$= \frac{2}{3-3} = \frac{2}{0} \rightarrow \infty$ Ans

Exp:- $\text{limit}_{x \to 0} \frac{x}{x-|x|}$

$= \lim_{x \to 0} \frac{x}{x-(x)}$

$= \lim_{x \to 0} \frac{\cancel{x}}{2\cancel{x}}$

$= \frac{1}{2}$ Ans

Exp:- $\lim_{h \to ?} \frac{|-1+h| - 1}{h}$

$\lim_{h \to 1} \frac{(-1+h) - 1}{h}$

$= \frac{(-1+1) - 1}{1} = -1$ Ans

Functional English

what is sentence

Sentence is a arrangement of words which means complete sense

Structure of sentence

A sentence has two parts

- subject
- predicate

Bracket function

Limit of Bracket function

Exp $[x](x+1)$

L.H.L $\lim_{x \to 0^-} [x](x+1)$

it approach to zero not actually equal to zero so

R.H.L $\lim_{x \to 0} [x](x+1)$

$= [0](0+1)$

Exp $\lim_{x \to 1} [2x](x \quad 1)$

L.H.L $\lim_{x \to 1} [2x](x \quad 1)$

0 1

$= 0$

Exp

$= 6$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ limit does not exist

Sandwich theorem:-

Exp

* [illegible]

$1 \le \lim_{x \to 0} x\left(\frac{1}{x}\right) \le 1$

By sandwich theorem

$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1$

Continuity

A function $f(x)$ is said to be continuous at $x = a \in R$ if

$\Rightarrow f(x)$ is defined at $x = a$

$\Rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$

Exp $g(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ [illegible] & \text{if } 1 \le x \le 10 \\ [illegible] & \text{if } x > 10 \end{cases}$

at $x = 1$, $x = 10$.

At $x = 1$

$f(x) = 4 - x^2$

$f(1) = 4 - (1)^2$

$= 5$

$LHL = \lim_{x \to 1^-} (x^2)$

$= 1$

R.H.L [illegible]

-5

$\therefore$ L.H.L $\neq$ R.H.L

So $g(x)$ is discontinous at $x = 1$

At $x = 10$

$g(x) = 4x^2$ [illegible]

L.H.L $\lim_{x \to 10}$ [illegible]

R.H.L $\lim_{x \to 10} (6x^2 + 46)$

$6(10)^2 + 46 \Rightarrow 646$

$g(x)$ is discontinous at $x = 10$

Exp $f(x) = \frac{x}{|x|}$ at $x = 0$

$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$

$\lim_{x \to 0^-} \frac{x}{x}$

$\lim_{x \to 0^+} \frac{x}{x} = 1$

This function is discontinuous [illegible]

Exp $g(x) = \begin{cases} \ldots & x \le 0 \\ \ldots & x > 0 \end{cases}$

at $x = 0$

$g(x) = 4$

$g(0) = \ldots$

LHL $= \lim\limits_{x \to 0^-} (\ldots)$

RHL $= \lim\limits_{x \to 0^+} |x-4|$

$= |0-4| = 4$

This function is discontinuous at $x=0$

Exp $g(x) = \begin{cases} \sin x & x \le \pi/4 \\ \cos x & x > \pi/4 \end{cases}$

at $x = \frac{\pi}{4}$

$f(x) = \sin x$

$= \sin \frac{\pi}{4}$

$= 1/\sqrt{2}$

LHL $= \lim\limits_{x \to \pi/4} (\sin x)$

$= \sin \frac{\pi}{4} = 1/\sqrt{2}$

RHL $= \lim\limits_{x \to \pi/4} \cos x$

$\cos \frac{\pi}{4} = 1/\sqrt{2}$

This function is continuous at $x = \pi/4$

Continuity

Exp $f(x) = \begin{cases} (1+3x)^{1/x} & \text{if } x \neq 0 \\ e^{2} & \text{if } x = 0 \end{cases}$

at $x = 0$.

[illegible]

$= \lim_{x \to 0} \left[(1+3x)^{1/3x} \right]^{3}$

$= e^{3}$

Hence $f(x)$ is discontinuous at $x = 0$

Exp $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^{2} & \text{if } x = 0 \end{cases}$

at $x = 0$

$f(0) = e^{2}$

$\lim_{x \to 0} (1+2x)^{1/x}$

$\lim_{x \to 0} \left((1+2x)^{1/2x} \right)^{2}$

$= e^{2}$

Hence $f(x)$ is continuous at $x = 0$

Exp $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} , & x \neq 0 \\ 1 , & x = 0 \end{cases}$

at $x = 0$

$f(0) = 1$

$L.H.L = \lim_{x \to 0^-} \dfrac{e^{1/x}}{1+e^{1/x}}$

$R.H.L = \lim_{x \to 0^+} \dfrac{e^{1/x}}{1+e^{1/x}}$

$= \dfrac{1}{\left(\frac{1}{e^{1/x}}+1\right)} = \dfrac{1}{0+1} = 1$

$L.H.L \neq R.H.L$ so $f(x)$ is discontinuous at $x=0$

$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Exp $= \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

if $x = 0$

$f(x) = 1$

$f(\ ) \quad 1$

$= \lim_{x \to 0} \frac{\sin 2x}{x}$

$= \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2$

= (when $x \to 0$

then

$2x \to 0$)

$= 2(1) = 2$

Practice * $f(x) = \begin{cases} \frac{\sin 2x}{\sin x} & \text{if } x \neq 0 \\ 2/\text{[illegible]} & \text{if } x = 0 \end{cases}$

* $f(x) = \begin{cases} x+4 & -6 < x < \text{[illegible]} \\ x & \text{[illegible]} < x < 2 \\ \text{[illegible]}x+1 & 2 < x < 6 \end{cases}$

Continuity:-

Points of Discontinuity

Exp

$$f(x) = \begin{cases} x^2 & \text{if } x<1 \\ -4-x^2 & \text{if } 1\le x\le 10 \\ 6x^2+46 & \text{if } x>10 \end{cases}$$

$f(x) = -4-x^2$

$f(1) = -4-(1)^2 = -5$

$L.H.L = \lim_{x\to 1^-} x^2$

$(1)^2 = 1$

$R.H.L = \lim_{x\to 1^+} -4-x^2$

$-4-(1)^2 = -5$

At $x=10$

$f(x) = -4-x^2$

$-4-(10)^2 = -104$

$\lim_{x\to 10} (-4-x^2)$

$-4-(10)^2 = -104$

$\lim_{x\to 10^+} (6x^2+46)$

$6(10)^2+46 = 646$

at $x=10$

Find the value of constant By the def of continuity

<u>Sol</u> $f(x) = \begin{cases} x^3 & \text{if } x \leq -1 \\ ax+b & \text{if } -1 \leq x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$

at $x = 1$

$f(x) = x^2+2$

$f(1) = 1+2 = 3$

$\lim (ax+b)$

$= a+b$

$\lim_{x \to 1} (x^2+2)$

$= 3$

[illegible]

at $x = -1$

$f(x) = x^3$

$= -1$

$\lim_{x \to -1} x^3$

$= -1$

$\lim_{x \to -1} (ax+b)$

$= a(-1)+b$

$= -a+b$

[illegible]

For MCQ

Find the interval of continuity

Ex: $f(x) = \frac{x^2-5}{x-1}$

Ex: $f(x) = \begin{cases} \frac{x^3-27}{x^2-9} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x=3$

$\lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{(x+3)(x-3)}$

Exp $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Derivative

Derivative of a function w.r.t. 'x' is denoted by $f'(x)$ and can be defined as

$$\lim_{\delta x \to} \frac{f(x+\delta x)+f(x)}{\delta x} = f'(x)$$

Derivative of f at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}.$$

L.H.D $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}.$

R.H.D $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}.$

Differentiable function

The function is said to be differentiable if its function exist.

Ex

Find the derivative of $f(x) = x^{2/3}$ at $x=0$

Sol: $f(x) = x^{2/3}$, $x=0$

$$f(0) = (0)^{2/3} = 0$$

$f'(0) = \lim_{x \to 0} \frac{x^{1/3} - 0}{x - 0}$

$= \lim_{x \to 0} \frac{x^{1/3}}{x}$

$= \lim_{x \to 0} x^{1/3 - 1}$

$= \lim_{x \to 0} x^{-1/3}$

$= \lim_{x \to 0} 1/x^{1/3}$

Apply limit

$= \frac{1}{(0)^{1/3}} = \frac{1}{0} \Rightarrow \infty$

Exp Find derivative of $f(x) = |x|$ at $x = 0$

$f(x) = |x|$

$f(0) = |0| = 0$

$f'(0) = \lim_{x \to 0} \frac{|x| - 0}{x - 0}$

$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$f'(0) = \lim_{x \to 0} \frac{|x|}{x} \longrightarrow ①$

$\lim_{x \to 0^-} \frac{|x|}{x} \Rightarrow \lim_{x \to 0^-} \frac{-x}{x} = -1$

$\lim_{x \to 0^+} \frac{|x|}{x} \Rightarrow \lim_{x \to 0^+} \frac{x}{x} = 1$

$L.H.L \neq R.H.L$

Hence ~~[illegible]~~ $f'(0)$ does not exist at $x = 0$.

Exp

Find

Ex $y = \ln 3x$

Find $f'(x)$ by def.

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \because \lim_{x \to 0} (1+x)^{1/x} = e$$

$$\because \ln a - \ln b = \ln \frac{a}{b}$$

$f(x) = \ln 3x$

$f(x+\delta x) = \ln 3(x+\delta x)$

Now

$$f'(x) = \lim_{\delta x \to 0} \frac{\ln 3(x+\delta x) - \ln 3x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln\left(\frac{\cancel{3}(x+\delta x)}{\cancel{3}x}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln\left(\frac{x+\delta x}{x}\right)}{\delta x} \Rightarrow \lim_{\delta x \to 0} \frac{\left(\frac{x}{x} + \frac{\delta x}{x}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\ln\left(1 + \frac{\delta x}{x}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) \qquad \because \ln a^b = b \ln a$$

multiply and divide by x.

$$= \lim_{\delta x \to 0} \frac{x}{\delta x \cdot x} \ln\left(1 + \frac{\delta x}{x}\right)$$

$$= \lim_{\delta x \to 0} \frac{1}{x} \left[\frac{x}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right)\right].$$

$$= \lim_{\delta x \to 0} \frac{1}{x} \ln\left(1 + \frac{\delta x}{x}\right)^{x/\delta x}$$

$$= \frac{1}{x} \ln\left[\lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{x/\delta x}\right.$$

$= \frac{1}{x} \ln e$ $\quad\quad$ $\because \ln e = 1$

$= \frac{1}{x}(1) = \frac{1}{x}$